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CRT-2002-12

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Centre for Research on Transportation

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May 2002

PUBLICATION
CRT-2002-12

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* This research was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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Dépôt légal – Bibliothèque nationale du Québec
Bibliothèque nationale du Canada, 2002

103-03065/

Abstract

In the General Routing Problem, the aim is to determine a least cost traversal of a subset of edges, arcs and vertices of a graph. The problem can be transformed into an equivalent Traveling Salesman Problem or Rural Postman Problem and solved optimally. Computational results are reported.

Key words : generalized routing problem, traveling salesman problem, rural postman problem, generalized traveling salesman problem.

Résumé

Dans le problème de tournées généralisé, on doit déterminer un parcours fermé de moindre coût sur un sous-ensemble d'arêtes, d'arcs et de sommets d'un graphe. Le problème se transforme en un problème du voyageur de commerce ou en un problème du postier rural équivalent. On peut le résoudre de façon optimale à l'aide de cette transformation. On présente des résultats numériques.

Mots-clefs : problème de tournées généralisé, problème du voyageur de commerce, problème du postier rural, problème du voyageur de commerce généralisé.

Introduction

The Generalized Routing Problem (GRP) is defined on a graph $G = (V, A \cup E)$,where $V = \{v_i\}$ is a set of vertices, A is a set of (directed) arcs, and E is a set of (undirected) edges. Non-negative costs c_{ij} and d_{ij} are associated with arcs (v_i, v_j) and with edges $[v_i, v_j]$, respectively. Denote by s_{ij} the length of a shortest path from i to j on G. The sets $V' \subseteq V, A' \subseteq A$ and $E' \subseteq E$ are required vertices, arcs and edges, respectively. If $A \neq \varnothing$ and $E = \varnothing$, the graph (and the GRP defined on it) is *directed* ; if $A \neq \varnothing$ and $E \neq \varnothing$, it is *undirected*. The GRP is to determine a minimum cost closed traversal of a subgraph of G including all required vertices, arcs and edges. This problem was introducted by Orloff[1] and shown to be NP-hard by Lenstra and Rinnooy Kan[2]. When V' is empty or contains only vertices incident to $A' \cup E'$, then the GRP reduces to the *Rural Postman Problem* (RPP)[2]. Recall that the RPP consists of determining a least cost traversal of all required arcs and edges of a graph. Here we assume without loss of generality that the vertices of V' are not incident to $A' \cup E'$.

The GRP arises in arc routing contexts where it is necessary not only to traverse some arcs or edges of a graph, corresponding to streets, but also some isolated vertices. Such problems are common in school bus routing where several children living on the same street must be collected by stopping at each house, but also groups of children who have to walk from their home to a specific bus stop. A similar situation arises in mail delivery problems where street segments and isolated buildings must be served.

Despite the practical importance of the GRP, relatively few studies have been published on this problem. Orloff proposed two heuristics to make the graph induced by V' and A' (or E') Eulerian by matching odd degree vertices, similar to what is done in the *Chinese Postman Problem* (see Edmonds and Johnson[3]), and then making the graph connected by branch-and-bound, as suggested by Bellmore and Malone[4]. For the mixed case, he suggested first replacing each required edge by a required vertex without providing further details. We believe this transformation is not valid since any transformation must be able to account for the direction in which an edge is traversed, and replacing it by a single vertex does not capture this information. Orloff provides no computational experience for any of his heuristics. Modifications to these heuristics were later proposed by Male, Liebman and Orloff[5] for the undirected case. In their article the authors wrongly claim that their procedure always yields an optimal solution. Orloff and Caprera[6] describe a heuristic in which sets of required vertices are first replaced by required edges and a feasible solution containing all required edges is then identified. Improvements are then sought by means of an edge interchange mechanism.

Much later, Jansen[7]proposed an approximation algorithm for undirected GRPs in which edge costs satisfy the triangle inequality. The procedure is based on the Christofides[8] approximation algorithm for the *Traveling Salesman Problem* (TSP)

and has a worst-case performance ratio of 3/2. Finally, Corberán, Letchford and Sanchis[9] have developed an exact branch-and-cut algorithm for the undirected GRP. The largest instance solved to optimality using this approach contains 200 vertices.

Our purpose is to show how several classes of the GRP can be solved exactly by transforming the problem into an equivalent vertex routing or arc routing problem for which an exact solution procedure is available. When the GRP is directed, it can readily be transformed into an asymmetric TSP. When it is mixed or undirected, it can be transformed into an asymmetric *Generalized Traveling Salesman Problem* (GTSP)[10], which can in turn be transformed into an asymmetric TSP[11]. The GRP can also be transformed into an equivalent RPP. Our choice of transformations is guided by the availability of exact algorithms for the transformed problem. The idea of using graph transformations to solve routing problems is not new. It was proposed, for example, by Lenstra and Rinnooy Kan[2] to transform the *m-Traveling Salesman Problem* into a standard TSP, and by Laporte[13] to transform several families of arc routing problems into an asymmetric TSP. The idea does not work equally well in all cases as the transformed graph can be rather degenerate, which can lead to excessive branching in the solution process. We found that it works best on directed problems and on mixed problems in which $|E|$ is relatively small.

In the next section, we will present the graph transformations and algorithms we proposed for the GRP, followed by computational results and by the conclusion.

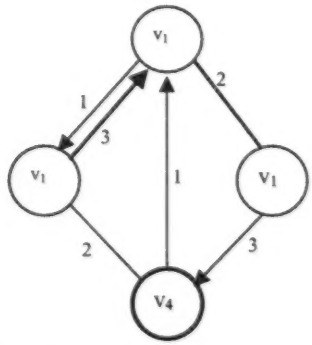
Graph transformations and algorithms

To transform a GRP on a graph $G = (V, A \cup E)$ into an equivalent vertex routing problem on a graph $\tilde{G} = (\tilde{V}, \tilde{A})$, we proceed as follows: 1) each vertex $v_i \in V'$ is relabeled v_{ii} ; 2) each arc $(v_i, v_j) \in A'$ is replaced by a required vertex v_{ij}; 3) each edge $[v_i, v_j] \in E'$ is first replaced by two opposite arcs (v_i, v_j) and (v_j, v_i) of cost $\tilde{c}_{ij} = \tilde{c}_{ji} = d_{ij}$. These arcs are then substituted by two vertices v_{ij} and v_{ji}, only one of which is required. The vertex set $\tilde{V}$ is then made up of all vertices v_{ij} just defined. Each vertex pair $(v_{ki}, v_{\ell j})$ in the transformed problem defines an arc of $\tilde{A}$ having a cost $\tilde{c}_{ij} = s_{i\ell} + c_{\ell j}$, where $\tilde{c}_{ii} = 0$. This transformation is depicted in Figure 1 for a mixed graph G.

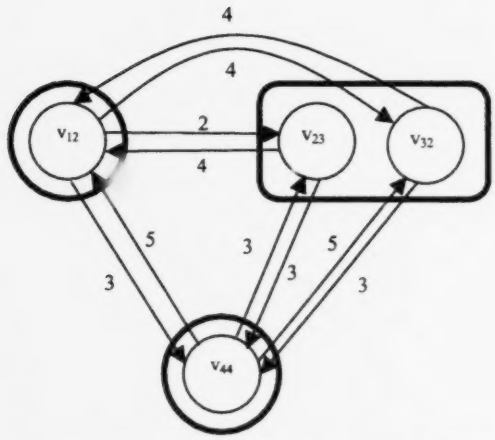
If $E' = \varnothing$, solving the asymmetric TSP on $\tilde{G}$ solves the GRP on G. If $E' \neq \varnothing$,the resulting problem on $\tilde{G}$ is not a TSP, but a GTSP since only one of the two vertices v_{ij} and v_{ji} coresponding to a required edge $[v_i, v_j]$ must appear in the solution. To obtain the appropriate GTSP, the vertex set $\tilde{V}$ is partitioned into clusters: define a cluster $\{v_{ii}\}$ for each $v_i \in V'$, and a cluster $\{v_{ij}, v_{ji}\}$ for each $[v_i, v_j] \in E'$. The GTSP then consists of determining on $\tilde{G}$, a minimum cost Hamiltonian circuit passing through each cluster exactly once. Solving this asymmetric GTSP on $\tilde{G}$ solves the original GRP on G. For a two-element cluster $\{v_{ij}, v_{ji}\}$ corresponding to an edge $[v_i, v_j]$ of E', if v_{ij} appears in the GTSP solution, then the edge is visited from v_i to v_j in the GRP; if v_{ji} appears in the GTSP solution, then the edge is visited from v_j to v_i.

Whether G is directed, mixed or undirected, the GRP can be transformed into an equivalent directed, mixed or undirected RPP if each required vertex $v_i \in V'$ is first replaced by a required edge $[v_i, v_j]$ which is incorporated into E'. We have only applied this procedure to the undirected GRP since we possessed an exact RPP algorithm for the undirected case, but not for the directed or mixed case.

The transformed problems are solved as follows. For the asymmetric TSP we use the Carpaneto, Dell'Amico and Toth[14,15] exact code which we believe to be the best available. The asymmetric GTSP is solved using two methods. A direct approach consists of applying to this problem the Lagrangean based branch-and-bound algorithm of Noon and Bean[10]. An undirected approach is to transform the asymmetric GTSP into an equivalent asymmetric TSP (involving the same number of vertices) as suggested by Noon and Bean[11] and then solving the resulting TSP by means of the Carpaneto, Dell'Amico and Toth algorithm. When each cluster contains only two vertices v_{ij} and v_{ji}, the transformation of the GTSP into an equivalent TSP is relatively simple. The two intra-cluster arcs are assigned a cost equal to an arbitrarily large negative constant $-M$; given any vertex $v_{k\ell}$ outside the cluster, the cost of arc $(v_{ji}, v_{k\ell})$ is substituted by that of arc $(v_{ji}, v_{k\ell})$ and the cost of arc $(v_{ji}, v_{k\ell})$ is substituted by that of arc $(v_{ij}, v_{k\ell})$. Because of the low cost of intra-cluster arcs, any TSP solution on the transformed graph will use the two v_{ij} and v_{ji} in succession. If v_{ji} precedes in the solution, this means edge $[v_i, v_j]$ is entered through v_i in the GRP defined on G The undirected RPP is solved using the exact branch-and-cut algorithm of Ghiani and Laporte[16].



a) GRP on a mixed graph G showing arc and edge costs. Bold components are required.



b) GTSP on the transformed directed graph $\widetilde{G}$.
The vertex set $\widetilde{V}$ is partitioned into three clusters.

Figure 1. Transformation of a GRP on a mixed graph into a directed GTSP on a directed graph

Computational results

The above transformations procedures were assessed on randomly generated directed, mixed and undirected graphs of various sizes, densities and proportions of required elements (vertices, arcs and edges). To ensure the feasiblity of each instance, we have first generated a Hamiltonian cycle over all vertices and we have included in the graphs all edges and arcs (in both directions) on this tour. Other arcs and edges were then added to the graph to reach a desired density. The required elements were also randomly determined. All arc and edge costs were randomly geenrated according to a discrete uniform distribution on $[10, 110]$.

All transformation procedures were coded in C. The Carpaneto, Dell'Amico and Toth algorithm for the asymmetric TSP and the Noon and Bean algorithm for the asymmetric GTSP, obtained from these authors, were coded in Fortran. The Ghiani and Laporte undirected RPP code is coded in C. All instances were solved on a Sun Ultra Sparc Station 10. We separately report the *transformation time* (TT) required for the shortest path computations and for the graph transformation, and the *resolution time* (RT) required to run the TSP, GTSP or RPP exact algorithm. No limit was imposed on TT but a limit of 300 seconds was set on RT. Five instances were generated for each parameter combination considered. Average statistics are reported for the *successful instances*, i.e., those for which RT did not exceed the limit.

Computational results are reported in Tables 1 to 3. The column headings are defined as follows:

$\|V\|$	:	number of vertices in G;
$\|A\|$	:	number of arcs in G;
$\|E\|$	:	number of edges in G;
$\|V'\|$	:	number of required vertices in G;
$\|A'\|$	:	number of required arcs in G;
$\|E'\|$	:	number of required edges in G;
$\|\tilde{V}\|$	:	number of vertices in $\tilde{G}$;
$\|\tilde{E}\|$	:	number of edges in $\tilde{G}$;
SUCC	:	number of successful instances out of 5;
TT	:	transformation time in seconds;
RT	:	resolution time in seconds.

In a first series of tests, we solved directed GRPs using the asymmetric TSP transformation and the Carpaneto, Dell'Amico and Toth algorithm. Results presented in Table 1 indicate that instances of this type can be solved optimally provided the number of vertices $|\tilde{V}| = |V'| + |A'|$ in the transformed graph does not exceed about 3500. The value of RT is affected by $|\tilde{V}|$ whereas the size of $|V|$ and $|A|$ only affect the shortest path computation times. The first two parts of the table show that RT increases with $|A'|$ for a fixed $|V'|$, with $|V'|$ for a fixed $|A'|$, whereas the third part shows that RT remains more or less the same for a constant $|\tilde{V}|$.

Table 1. Summary of computational results for directed GRPs transformed into an asymmetric TSP and solved by the Carpaneto, Dell'Amico and Toth algorithm

$\|V\|$	$\|A\|$	$\|V'\|$	$\|A'\|$	$\|\tilde{V}\|$	SUCC	TT	RT
5000	50000	1000	1000	2000	5	73.1	52.5
5000	50000	1000	1500	2500	5	93.2	100.4
5000	50000	1000	2000	3000	5	111.9	168.4
5000	50000	1000	2500	3500	4	129.2	245.7
5000	50000	1000	3000	4000	0	-	-
5000	50000	1000	1000	2000	5	73.1	52.8
5000	50000	1500	1000	2500	5	89.6	93.5
5000	50000	2000	1000	3000	5	107.0	137.7
5000	50000	2500	1000	3500	5	125.7	188.8
5000	50000	3000	1000	4000	4	144.3	252.5
5000	50000	0	3000	3000	5	112.1	190.9
5000	50000	500	2500	3000	5	112.3	187.7
5000	50000	1000	2000	3000	5	111.5	167.6
5000	50000	1500	1500	3000	5	112.7	156.5
5000	50000	2000	1000	3000	5	107.8	138.0
5000	50000	2500	500	3000	5	105.4	120.7
5000	50000	3000	0	3000	4	95.8	178.1

In a second series of tests, we solved mixed GRPs transformed into asymmetric GTSPs. These GTSPs were either solved directly using the Noon and Bean algorithm, or transformed into an asymmetric TSP and solved by the Carpaneto, Dell'Amico and Toth algorithm. Computational results are summarized in Table 2. These results show that the mixed GRP is much more difficult than the directed version. Problem difficulty is highly related to the size of $|E'|$ which determines the number of clusters in the GTSP. The size of $|A'|$ is less critical. For instances yielding the same value of $|\tilde{V}|$ in $\tilde{G}$ (third part of the table), computational difficulty seems more related to $|V'|$ than to $|A'|$. We solved instances involving up to 25 required edges in G and up to 230 vertices in $\tilde{G}$. Generally, first transforming GTSPs in TSPs works better than solving them directly, except when $|A'|$ is small relative to $|E'|$ (first part of the table). It should be noted that the TSPs derived from the GTSP are more difficult than TSPs with random cost or obtained from a directed GRP instance. This due to the fact that the TSPs derived from a mixed GRP contain several symmetrical costs between pairs of vertices, leading to the creation of two-vertex subtours and considerable branching.

Table 2. Summary of computational results for mixed GRPs transformed into an asymmetric GTSP and solved directly (Noon and Bean) or using a transformation into an asymmetric TSP (Carpaneto, Dell'Amico and Toth)

							Direct			Asymmetric TSP		
$\|V\|$	$\|A\|$	$\|E\|$	$\|V'\|$	$\|A'\|$	$\|E'\|$	$\|\tilde{V}\|$	SUCC	TT	TR	SUCC	TT	TR
500	1000	1000	10	10	5	30	5	0.1	0.1	5	0.0	0.0
500	1000	1000	10	10	10	40	5	0.1	0.3	5	0.0	1.1
500	1000	1000	10	10	15	50	5	0.2	1.3	4	0.0	112.4
500	1000	1000	10	10	20	60	5	0.2	11.3	0	-	-
500	1000	1000	10	10	25	70	5	0.2	5.7	0	-	-
500	1000	1000	10	10	10	40	5	0.1	0.3	5	0.0	1.0
500	1000	1000	50	50	10	120	5	0.4	80.9	5	0.1	17.9
500	1000	1000	50	100	10	170	4	0.6	96.0	5	0.3	36.8
500	1000	1000	100	50	10	170	1	0.6	109.6	5	0.3	43.6
500	1000	1000	10	200	10	230	5	1.0	83.4	5	0.5	20.3
500	1000	1000	0	110	10	130	5	0.5	7.5	5	0.2	6.5
500	1000	1000	30	80	10	130	5	0.5	10.8	5	0.2	9.1
500	1000	1000	55	55	10	130	4	0.5	34.5	5	0.2	12.5
500	1000	1000	80	30	10	130	3	0.4	121.1	5	0.2	48.1
500	1000	1000	110	0	10	130	0	-	-	0	-	-

In a third and last series of tests, we considered undirected GRPs transformed into undirected RPPs and solved exactly by means of the Ghiani and Laporte algorithm. We also transformed the GRP instances into GTSPs, but this approach

produced rather disappointing results and was abandoned. As can be seen from Table 3, problem difficulty is again related to the size of $|V'|$ and $|E'|$, but for a fixed value of $|\overline{E}|$ instances with a large $|V'|$ tend to be more difficult (third part of the table). We successfully solved instances containing up to 100 required elements. A direct comparison with the algorithm of Corberán, Letchford and Sanchis[9] is difficult to make since these authors did not report their statistics in the same way as we did. However, they solved instances containing up to 200 vertices in V whereas ours have up to 500 in V and 590 in $\tilde{V}$. On the whole, we seem to be solving much larger instances faster.

Table 3 : Summary of computational results for undirected GRPs transformed into an undirected RPP and solved by the Ghiani and Laporte algorithm

$\|V\|$	$\|E\|$	$\|V'\|$	$\|E'\|$	$\|\tilde{V}\|$	$\|\tilde{E}\|$	$\|\tilde{E}'\|$	SUCC	TT	RT
500	2000	10	50	510	2010	60	5	0.0	11.2
500	2000	20	50	520	2020	70	5	0.0	24.4
500	2000	30	50	530	2030	80	5	0.0	46.0
500	2000	40	50	540	2040	90	3	0.0	62.0
500	2000	50	50	550	2050	100	5	0.0	103.8
500	2000	50	10	550	2050	60	5	0.0	18.4
500	2000	50	20	550	2050	70	5	0.0	38.6
500	2000	50	30	550	2050	80	5	0.0	42.0
500	2000	50	40	550	2050	90	4	0.0	101.2
500	2000	50	50	550	2050	100	5	0.0	104.6
500	2000	10	90	510	2010	100	5	0.0	61.4
500	2000	30	70	530	2030	100	5	0.0	84.6
500	2000	50	50	550	2050	100	5	0.0	101.6
500	2000	70	30	570	2070	100	4	0.0	108.5
500	2000	90	10	590	2090	100	2	0.0	135.5

Conclusion

We have provided and tested transformation procedures for the Generalized Routing Problem. This problem arises in a number of distribution and arc routing settins where segments as well as isolated locations must be served. The transformation are easy to implement. Their main advantage is that efficient algorithms designed for the transformed problem can be applied directly, as a black box, without any modification. For almost all the cases considered, this is the only available methodology. For the undirected case where an alternative exact algorithm exists, our approach seems capable of handling larger instances.

Acknowledgements

This work was partly supported by the Canadian Natural Sciences and Engineering Research Council under grant OGP0039682. This support is gratefully acknowledges. Thanks are also due to Charles E. Noon and Paolo Toth who kindly supplied their GTSP and TSP computer codes. This work was partly carried out while the second author visited the Centre for Traffic and Transport at the Danish Technical University.

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